Tentamen: Introduction to Plasma Physics

January 19, 2010

9.00-12.00 h, Room 5116.0116

Please write clearly your name on *each* sheet, and on the first sheet also your student number, date of birth, and address. You can use either dutch or english. Relevant plasma formulas are listed on the last page.

PROBLEM 1 (15 points)

A plasma has several characteristic parameters such as the Debye length λ_{dB}, the electron plasma frequency ν_{pe}, the electron cyclotron frequency ν_{ce} and the electron-electron collision frequency ν_{ee}.

a. Give a brief description of the physical meaning and significance of each parameter.

b. Derive the formula for the electron cyclotron frequency ν_{ce} in terms of the fundamental parameters of the electron.

c. Consider a fully ionized gas with equal electron and ion density $n = 10^{15}$ cm^{-3}. The plasma temperature T is 0.1 eV. Does this ionized gas behave as a plasma? Motivate your answer.

d. Can an electromagnetic wave with a frequency $\nu = 14$ GHz propagate through this plasma? Motivate your answer.

e. Does this plasma qualify as a collisionless plasma? Motivate your answer.

PROBLEM 2 (25 points)

A charged particle of mass m and charge q has its guiding center at time $t = 0$ at a distance r_0 from the center of a long straight wire carrying a current I_0. A uniform electric field E_0 exists parallel to the axis of the wire. The velocities parallel and perpendicular to the magnetic field lines are denoted as $v_\parallel$ and $v_\perp$, respectively.

a. Briefly explain the concept of the guiding center and its usefulness in understanding the motion of charged particles in inhomogeneous magnetic fields. Mention two different circumstances under which the guiding center concept breaks down and explain why.

b. Use Ampère's law to show that the magnetic field of the current-carrying wire is given by

$$\mathbf{B} = \frac{\mu_0 I_0}{2\pi r}\,\mathbf{e}_\theta\,,$$

with r the distance from the wire and $\mathbf{e}_\theta$ a unit vector in the azimuthal θ direction.

c. Determine the magnitudes and directions of the various drift velocities, i.e. the $\mathbf{E} \times \mathbf{B}$ drift velocity, the gradient-B drift velocity, and the curvature-B drift velocity in terms of the parameters I_0 and E_0. Use cylindrical coordinates (r, θ, z).

d. Calculate the r component of the guiding center position as a function of time: $r_{gc}(t)$.

e. Draw the guiding center trajectory in a schematic diagram.

PROBLEM 3 (25 points)

Consider a cylindrically symmetric plasma with radius a confined in a theta pinch as shown in Figure 1. The radius of the plasma chamber wall is b with $b > a$. A purely azimuthal current I in a cylindrical coil produces a uniform axial vacuum magnetic field B_0 in the region between the plasma and the cylindrical plasma chamber wall $a < r < b$. The induced current density in the plasma is in the azimuthal direction and given by $J_\theta(r) = Ar(a - r)$, the resulting axial magnetic field in the plasma column is $B_z(r)$. The radial plasma pressure distribution is $p(r)$.

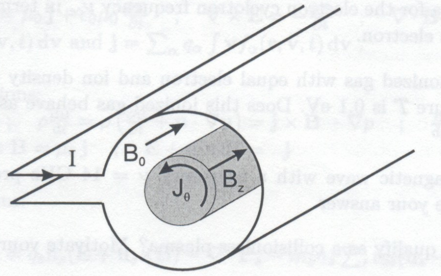

Figure 1: Theta pinch with a cylindrically symmetric plasma with pressure profile $p(r)$ and axial magnetic field $B_z(r)$. The axial vacuum magnetic field B_0 for $a < r < b$ is uniform.

a. Derive the one-dimensional equilibrium MHD force balance equation:

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla)\mathbf{B}$$

and give an interpretation of each term.

b. Apply the above MHD force balance equation to the theta pinch to obtain an equation for $p(r)$, $B_z(r)$ and B_0.

c. Assume that the total induced current in the plasma is I_p. Calculate the normalization constant A in the expression $J_\theta(r) = Ar(a - r)$.

d. Calculate the magnetic field inside the plasma $B_z(r)$.

e. Calculate the radial pressure profile inside the plasma $p(r)$. If you did not get an answer for $B_z(r)$ assume that $B_z(r) = B_0(3r^2/2a^2 - r^3/a^3 + 1/2)$.

f. Draw the radial profiles $p(r)$, $B_z(r)$ and $J_\theta(r)$ for $0 < r < b$ in a schematic diagram.

PROBLEM 4 (25 points)

Consider a streaming instability in a collisionless plasma in which the electrons with mass m_e move with a velocity $\mathbf{v}_0$ through stationary ions with mass m_i. There is no magnetic field and both the ions and electrons are cold, i.e. $T_i = T_e = 0$. You are asked to derive the dispersion relation for small-amplitude electrostatic waves propagating along the direction of the streaming electrons and analyze the stability of these waves using the two-fluid MHD equations. Assume that the undisturbed plasma is uniform and neutral, i.e. $n_{i0} = n_{e0} = n_0 =$ const.

a. First briefly explain what a dispersion relation is and what its relevance is for studying wave propagation.

b. Write down the linearized momentum-balance equations for the ions and electrons, i.e. insert $n_i = n_0 + n_{i1}$, $n_e = n_0 + n_{e1}$, $\mathbf{v}_i = \mathbf{v}_{i1}$ and $\mathbf{v}_e = \mathbf{v}_0 + \mathbf{v}_{i1}$ and derive the first-order momentum-balance equations.

c. Assume that the first-order perturbations vary as $e^{i(kz-\omega t)}$ and transform the first-order momentum-balance equations into algebraic equations. Calculate the first-order velocities v_{i1} and v_{e1} of the ions and electrons (these velocities are in the z-direction).

d. Do the same with the continuity equation and derive the first-order ion and electron densities n_{i1} and n_{e1}.

e. Fourier transform Gauss's law and show that the dispersion relation is given by

$$1 = \omega_{pe}^2 \left[\frac{m_e/m_i}{\omega^2} + \frac{1}{(\omega - v_0)^2} \right]$$

f. Explain how you can graphically solve this dispersion relation. What can you say about the stability of electrostatic waves that propagate along the electron beam.

3

Vector double cross product:
$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Characteristic plasma parameters:
$\lambda_{dB} = \sqrt{\frac{\epsilon_o k_B T_e}{n_e e^2}} = 743\sqrt{\frac{T_e(\mathrm{eV})}{n_e(\mathrm{cm}^{-3})}}\ (\mathrm{cm})$; $\nu_{pe} = \frac{\omega_{p,e}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{e^2 n_e}{m_e\epsilon_o}} = 8.98\sqrt{n_e(\mathrm{cm}^{-3})}\ (\mathrm{kHz})$
$\nu_{ce}(\mathrm{GHz}) = 28\,B(\mathrm{T})$; $\nu_{ee}(\mathrm{s}^{-1}) = 4\cdot 10^{-12}\frac{n_e(\mathrm{m}^{-3})\ln\Lambda}{T_e^{3/2}(\mathrm{eV})}$

Guiding center drifts:
$\mathbf{v}_{gc,E\times B} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$; $\mathbf{v}_{gc,\nabla B} = \frac{m v_\perp^2}{2qB^3}\mathbf{B}\times\nabla B$; $\mathbf{v}_{gc,cB} = \frac{m v_\parallel^2}{qB^3}\mathbf{B}\times\nabla B$

Boltzmann equation: $\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}[\mathbf{E}+\mathbf{v}\times\mathbf{B}]\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$

Electro- and magnetostatics:
$\epsilon_0\oint_S \mathbf{E}\cdot\mathbf{n}\,d\sigma = Q_{encl}$; $\oint_C \mathbf{B}\cdot\mathbf{t}\,ds = \mu_0 I_{encl}$

Maxwell's equations:
$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$; $\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}$; $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$; $\nabla\cdot\mathbf{B} = 0$
with $\rho = \sum_\alpha q_\alpha\int f_\alpha(\mathbf{r},\mathbf{v},t)\,d\mathbf{v}$ and $\mathbf{j} = \sum_\alpha q_\alpha\int \mathbf{v} f_\alpha(\mathbf{r},\mathbf{v},t)\,d\mathbf{v}$

Single-fluid MHD equations:
$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$; $\rho\frac{d\mathbf{u}}{dt} = \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \mathbf{j}\times\mathbf{B} - \nabla p$; $\frac{d}{dt}(p\,\rho^{-\gamma}) = 0$
$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$; $\nabla\times\mathbf{B} = \mu_0\mathbf{j}$; $\mathbf{E} + \mathbf{u}\times\mathbf{B} = \mathbf{j}$

Two-fluid MHD equations:
$\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\mathbf{u}_s) = 0$
$m_s n_s\left[\frac{\partial\mathbf{u}_s}{\partial t} + (\mathbf{u}_s\cdot\nabla)\mathbf{u}_s\right] = q_s n_s(\mathbf{E} + \mathbf{u}_s\times\mathbf{B}) - \nabla\cdot\mathbf{P}_s - m_s n_s\sum_t \nu_{st}(\mathbf{u}_s - \mathbf{u}_t)$